1. Consider the system of equations:

$$\begin{array}{rcrcrcr} 9x_1 & + & 2x_2 & - & x_3 & = & -2 \\ -x_1 & + & 6x_2 & - & x_3 & = & 2 \\ 2x_1 & - & x_2 & + & 8x_3 & = & 3 \end{array}$$

Solve this system to four-digit accuracy using the Jacobi algorithm.

solution:

Start by moving all the off-diagonal terms to the right-hand side, then divide by the diagonal coefficients, to yield:

$$\begin{array}{rclcrcrcr} x_1 & = & & - & \frac{2}{9}x_2 & + & \frac{1}{9}x_3 & - & \frac{2}{9} \\ x_2 & = & \frac{1}{6}x_1 & & & + & \frac{1}{6}x_3 & + & \frac{1}{3} \\ x_3 & = & -\frac{1}{4}x_1 & + & \frac{1}{8}x_2 & & & + & \frac{3}{8} \end{array}$$

So the Jacobi algorithm for this problem becomes:

$$\begin{array}{rclcrcrcr} x_1^{(k+1)} & = & & - & \frac{2}{9}x_2^{(k)} & + & \frac{1}{9}x_3^{(k)} & - & \frac{2}{9} \\ x_2^{(k+1)} & = & \frac{1}{6}x_1^{(k)} & & & + & \frac{1}{6}x_3^{(k)} & + & \frac{1}{3} \\ x_3^{(k+1)} & = & -\frac{1}{4}x_1^{(k)} & + & \frac{1}{8}x_2^{(k)} & & & + & \frac{3}{8} \end{array}$$

Starting out with:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1^{(0)} = 0 \ , \quad x_2^{(0)} = 0 \ , \quad x_3^{(0)} = 0$$

and so

$$\begin{array}{rclcrcrcr} x_1^{(1)} & = & & - & \frac{2}{9}x_2^{(0)} & + & \frac{1}{9}x_3^{(0)} & - & \frac{2}{9} \\ x_2^{(1)} & = & \frac{1}{6}x_1^{(0)} & & & + & \frac{1}{6}x_3^{(0)} & + & \frac{1}{3} \\ x_3^{(1)} & = & -\frac{1}{4}x_1^{(0)} & + & \frac{1}{8}x_2^{(0)} & & & + & \frac{3}{8} \end{array}$$

or

$$\begin{array}{rclcrcrcrcr} x_1^{(1)} & = & & - & \frac{2}{9}(0) & + & \frac{1}{9}(0) & - & \frac{2}{9} & = & -\frac{2}{9} \\ x_2^{(1)} & = & \frac{1}{6}(0) & & & + & \frac{1}{6}(0) & + & \frac{1}{3} & = & \frac{1}{3} \\ x_3^{(1)} & = & -\frac{1}{4}(0) & + & \frac{1}{8}(0) & & & + & \frac{3}{8} & = & \frac{3}{8} \end{array}$$

solution:

or, in terms of four-digit numbers

$$\mathbf{x}^{(1)} = \begin{bmatrix} -0.2222 \\ 0.3333 \\ 0.3750 \end{bmatrix}$$

Continuing:

$$\begin{aligned}
x_1^{(2)} &= & &- \tfrac{2}{9}(0.3333) &+ \tfrac{1}{9}(0.3750) &- \tfrac{2}{9} &= -0.2546 \\
x_2^{(2)} &= \tfrac{1}{6}(-0.2222) & & &+ \tfrac{1}{6}(0.3750) &+ \tfrac{1}{3} &= 0.3588 \\
x_3^{(2)} &= -\tfrac{1}{4}(-0.2222) & &+ \tfrac{1}{8}(0.3333) & &+ \tfrac{3}{8} &= 0.4722
\end{aligned}$$

and

$$\begin{aligned}
x_1^{(3)} &= & &- \tfrac{2}{9}(0.3588) &+ \tfrac{1}{9}(0.4722) &- \tfrac{2}{9} &= -0.2495 \\
x_2^{(3)} &= \tfrac{1}{6}(-0.2546) & & &+ \tfrac{1}{6}(0.4722) &+ \tfrac{1}{3} &= 0.3696 \\
x_3^{(3)} &= -\tfrac{1}{4}(-0.2546) & &+ \tfrac{1}{8}(0.3588) & &+ \tfrac{3}{8} &= 0.4835
\end{aligned}$$

Repeated application yields:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	−0.2222	0.3333	0.3750
2	−0.2546	0.3588	0.4722
3	−0.2495	0.3696	0.4835
4	−0.2506	0.3723	0.4836
5	−0.2512	0.3722	0.4842
6	−0.2511	0.3722	0.4843
7	−0.2511	0.3722	0.4843

So the method converges (to four significant digits) in six iterations.

2. Resolve problem 1, again to four-digit accuracy, using the Gauss-Seidel algorithm.

solution:

Start by moving all the above-diagonal terms to the right-hand side, then divide by the diagonal coefficients, to yield:

$$\begin{array}{rcrcrcrcrcr} x_1 & & & & & = & - & \frac{2}{9}x_2 & + & \frac{1}{9}x_3 & - & \frac{2}{9} \\ -\frac{1}{6}x_1 & + & x_2 & & & = & & & + & \frac{1}{6}x_3 & + & \frac{1}{3} \\ \frac{1}{4}x_1 & - & \frac{1}{8}x_2 & + & x_3 & = & & & & & + & \frac{3}{8} \end{array}$$

So the Gauss-Seidel algorithm for this problem becomes:

$$\begin{array}{rcrcrcrcrcr} x_1^{(k+1)} & & & & & = & - & \frac{2}{9}x_2^{(k)} & + & \frac{1}{9}x_3^{(k)} & - & \frac{2}{9} \\ -\frac{1}{6}x_1^{(k+1)} & + & x_2^{(k+1)} & & & = & & & + & \frac{1}{6}x_3^{(k)} & + & \frac{1}{3} \\ \frac{1}{4}x_1^{(k+1)} & - & \frac{1}{8}x_2^{(k+1)} & + & x_3^{(k+1)} & = & & & & & + & \frac{3}{8} \end{array}$$

or

$$\begin{array}{rcrcrcrcr} x_1^{(k+1)} & = & & - & \frac{2}{9}x_2^{(k)} & + & \frac{1}{9}x_3^{(k)} & - & \frac{2}{9} \\ x_2^{(k+1)} & = & \frac{1}{6}x_1^{(k+1)} & & & + & \frac{1}{6}x_3^{(k)} & + & \frac{1}{3} \\ x_3^{(k+1)} & = & -\frac{1}{4}x_1^{(k+1)} & + & \frac{1}{8}x_2^{(k+1)} & & & + & \frac{3}{8} \end{array}$$

Starting out with:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1^{(0)} = 0\ , \quad x_2^{(0)} = 0\ , \quad x_3^{(0)} = 0$$

and so

$$\begin{array}{rcrcrcrcr} x_1^{(1)} & = & & - & \frac{2}{9}x_2^{(0)} & + & \frac{1}{9}x_3^{(0)} & - & \frac{2}{9} \\ x_2^{(1)} & = & \frac{1}{6}x_1^{(1)} & & & + & \frac{1}{6}x_3^{(0)} & + & \frac{1}{3} \\ x_3^{(1)} & = & -\frac{1}{4}x_1^{(1)} & + & \frac{1}{8}x_2^{(1)} & & & + & \frac{3}{8} \end{array}$$

or, using four-digit numbers,

$$\begin{array}{rcrcrcrcrcr} x_1^{(1)} & = & & - & \frac{2}{9}(0) & + & \frac{1}{9}(0) & - & \frac{2}{9} & = & -0.2222 \\ x_2^{(1)} & = & \frac{1}{6}(-0.2222) & & & + & \frac{1}{6}(0) & + & \frac{1}{3} & = & 0.2963 \\ x_3^{(1)} & = & -\frac{1}{4}(-0.2222) & + & \frac{1}{8}(0.2963) & & & + & \frac{3}{8} & = & 0.4676 \end{array}$$

solution:

or, in terms of four-digit numbers

$$\mathbf{x}^{(1)} = \begin{bmatrix} -0.2222 \\ 0.2963 \\ 0.4676 \end{bmatrix}$$

Continuing:

$$\begin{aligned}
x_1^{(2)} &= & & - \tfrac{2}{9}(0.2963) & + \tfrac{1}{9}(0.4676) & - \tfrac{2}{9} & = -0.2361 \\
x_2^{(2)} &= & \tfrac{1}{6}(-0.2361) & & + \tfrac{1}{6}(0.4676) & + \tfrac{1}{3} & = 0.3719 \\
x_3^{(2)} &= & -\tfrac{1}{4}(-0.2361) & + \tfrac{1}{8}(0.3719) & & + \tfrac{3}{8} & = 0.4805
\end{aligned}$$

and

$$\begin{aligned}
x_1^{(3)} &= & & - \tfrac{2}{9}(0.3719) & + \tfrac{1}{9}(0.4805) & - \tfrac{2}{9} & = -0.2515 \\
x_2^{(3)} &= & \tfrac{1}{6}(-0.2515) & & + \tfrac{1}{6}(0.4805) & + \tfrac{1}{3} & = 0.3715 \\
x_3^{(3)} &= & -\tfrac{1}{4}(-0.2515) & + \tfrac{1}{8}(0.3715) & & + \tfrac{3}{8} & = 0.4843
\end{aligned}$$

Repeated application yields:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	-0.2222	0.2963	0.4676
2	-0.2361	0.3719	0.4805
3	-0.2515	0.3715	0.4843
4	-0.2510	0.3722	0.4843
5	-0.2511	0.3722	0.4843
6	-0.2511	0.3722	0.4843

So the method converges (to four significant digits) in five iterations.

3. Consider the system of equations:

$$\begin{array}{rcrcrcr} 10x_1 & - & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & - & 8x_2 & + & x_3 & = & 1 \\ x_1 & - & x_2 & + & 8x_3 & = & -3 \end{array}$$

Solve this system to four-digit accuracy using the Jacobi algorithm.

solution:

Start by moving all the off-diagonal terms to the right-hand side, then divide by the diagonal coefficients, to yield:

$$\begin{array}{rcrcrcrcr} x_1 & = & & & \frac{2}{10}x_2 & - & \frac{1}{10}x_3 & + & \frac{1}{10} \\ x_2 & = & \frac{2}{8}x_1 & & & + & \frac{1}{8}x_3 & - & \frac{1}{8} \\ x_3 & = & -\frac{1}{8}x_1 & + & \frac{1}{8}x_2 & & & - & \frac{3}{8} \end{array}$$

So the Jacobi algorithm for this problem becomes:

$$\begin{array}{rcrcrcrcr} x_1^{(k+1)} & = & & & \frac{2}{10}x_2^{(k)} & - & \frac{1}{10}x_3^{(k)} & + & \frac{1}{10} \\ x_2^{(k+1)} & = & \frac{2}{8}x_1^{(k)} & & & + & \frac{1}{8}x_3^{(k)} & - & \frac{1}{8} \\ x_3^{(k+1)} & = & -\frac{1}{8}x_1^{(k)} & + & \frac{1}{8}x_2^{(k)} & & & - & \frac{3}{8} \end{array}$$

Starting out with:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1^{(0)} = 0 \ , \quad x_2^{(0)} = 0 \ , \quad x_3^{(0)} = 0$$

and so

$$\begin{array}{rcrcrcrcr} x_1^{(1)} & = & & & \frac{2}{10}x_2^{(0)} & - & \frac{1}{10}x_3^{(0)} & + & \frac{1}{10} \\ x_2^{(1)} & = & \frac{2}{8}x_1^{(0)} & & & + & \frac{1}{8}x_3^{(0)} & - & \frac{1}{8} \\ x_3^{(1)} & = & -\frac{1}{8}x_1^{(0)} & + & \frac{1}{8}x_2^{(0)} & & & - & \frac{3}{8} \end{array}$$

or

$$\begin{array}{rcrrrrcr} x_1^{(1)} & = & & \frac{2}{10}(0) & -\frac{1}{10}(0) & +\frac{1}{10} & = & \frac{1}{10} \\ x_2^{(1)} & = & \frac{2}{8}(0) & & +\ \frac{1}{8}(0) & -\frac{1}{8} & = & -\frac{1}{8} \\ x_3^{(1)} & = & -\frac{1}{8}(0) & +\frac{1}{8}(0) & & -\frac{3}{8} & = & -\frac{3}{8} \end{array}$$

solution:

or, in terms of four-digit numbers

$$\mathbf{x}^{(1)} = \begin{bmatrix} 0.1000 \\ -0.1250 \\ -0.3750 \end{bmatrix}$$

Continuing:

$$\begin{aligned}
x_1^{(2)} &= \tfrac{2}{10}(-0.1250) - \tfrac{1}{10}(-0.3750) + \tfrac{1}{10} = 0.1125 \\
x_2^{(2)} &= \tfrac{2}{8}(0.1000) + \tfrac{1}{8}(-0.3750) - \tfrac{1}{8} = -0.1469 \\
x_3^{(2)} &= -\tfrac{1}{8}(0.1000) + \tfrac{1}{8}(-0.1250) - \tfrac{3}{8} = -0.4031
\end{aligned}$$

and

$$\begin{aligned}
x_1^{(3)} &= \tfrac{2}{10}(-0.1469) - \tfrac{1}{10}(-0.4031) + \tfrac{1}{10} = 0.1109 \\
x_2^{(3)} &= \tfrac{2}{8}(0.1125) + \tfrac{1}{8}(-0.4031) - \tfrac{1}{8} = -0.1473 \\
x_3^{(3)} &= -\tfrac{1}{8}(0.1125) + \tfrac{1}{8}(-0.1469) - \tfrac{3}{8} = -0.4074
\end{aligned}$$

Repeated application yields:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0.1000	−0.1250	−0.3750
2	0.1125	−0.1469	−0.4031
3	0.1109	−0.1473	−0.4074
4	0.1113	−0.1482	−0.4073
5	0.1111	−0.1481	−0.4074
6	0.1111	−0.1482	−0.4074
7	0.1111	−0.1482	−0.4074

So the method converges (to four significant digits) in six iterations.

4. Resolve problem 3, again to four-digit accuracy, using the Gauss-Seidel algorithm.

solution:

Start by moving all the above-diagonal terms to the right-hand side, then divide by the diagonal coefficients, to yield:

$$\begin{array}{rcrcrcrcrcr} x_1 & & & & & = & \frac{2}{10}x_2 & - & \frac{1}{10}x_3 & + & \frac{1}{10} \\ -\frac{2}{8}x_1 & + & x_2 & & & = & & + & \frac{1}{8}x_3 & - & \frac{1}{8} \\ \frac{1}{8}x_1 & - & \frac{1}{8}x_2 & + & x_3 & = & & & & - & \frac{3}{8} \end{array}$$

So the Gauss-Seidel algorithm for this problem becomes:

$$\begin{array}{rcrcrcrcrcr} x_1^{(k+1)} & & & & & = & \frac{2}{10}x_2^{(k)} & - & \frac{1}{10}x_3^{(k)} & + & \frac{1}{10} \\ -\frac{2}{8}x_1^{(k+1)} & + & x_2^{(k+1)} & & & = & & + & \frac{1}{8}x_3^{(k)} & - & \frac{1}{8} \\ \frac{1}{8}x_1^{(k+1)} & - & \frac{1}{8}x_2^{(k+1)} & + & x_3^{(k+1)} & = & & & & - & \frac{3}{8} \end{array}$$

or

$$\begin{array}{rcrcrcrcrcr} x_1^{(k+1)} & = & & & \frac{2}{10}x_2^{(k)} & - & \frac{1}{10}x_3^{(k)} & + & \frac{1}{10} \\ x_2^{(k+1)} & = & \frac{2}{8}x_1^{(k+1)} & & & + & \frac{1}{8}x_3^{(k)} & - & \frac{1}{8} \\ x_3^{(k+1)} & = & -\frac{1}{8}x_1^{(k+1)} & + & \frac{1}{8}x_2^{(k+1)} & & & - & \frac{3}{8} \end{array}$$

Starting out with:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad x_1^{(0)} = 0 \ , \quad x_2^{(0)} = 0 \ , \quad x_3^{(0)} = 0$$

implies

$$\begin{array}{rcrcrcrcrcr} x_1^{(1)} & = & & & \frac{2}{10}x_2^{(0)} & - & \frac{1}{10}x_3^{(0)} & + & \frac{1}{10} \\ x_2^{(1)} & = & \frac{2}{8}x_1^{(1)} & & & + & \frac{1}{8}x_3^{(0)} & - & \frac{1}{8} \\ x_3^{(1)} & = & -\frac{1}{8}x_1^{(1)} & + & \frac{1}{8}x_2^{(1)} & & & - & \frac{3}{8} \end{array}$$

or, using four-digit numbers,

$$\begin{array}{rcrcrcrcrcr} x_1^{(1)} & = & & - & \frac{2}{10}(0) & + & \frac{1}{10}(0) & + & \frac{1}{10} & = & 0.1000 \\ x_2^{(1)} & = & \frac{2}{8}(0.1000) & & & + & \frac{1}{8}(0) & - & \frac{1}{8} & = & -0.1000 \\ x_3^{(1)} & = & -\frac{1}{8}(0.1000) & + & \frac{1}{8}(-0.1000) & & & - & \frac{3}{8} & = & -0.4000 \end{array}$$

solution:

or, in terms of four-digit numbers

$$\mathbf{x}^{(1)} = \begin{bmatrix} 0.1000 \\ -0.1000 \\ -0.4000 \end{bmatrix}$$

Continuing:

$$\begin{aligned}
x_1^{(2)} &= & & - \tfrac{2}{10}(-0.1000) & &+ \tfrac{1}{10}(-0.4000) & &+ \tfrac{1}{10} = 0.1200 \\
x_2^{(2)} &= \tfrac{2}{8}(0.1200) & & & &+ \tfrac{1}{8}(-0.4000) & &- \tfrac{1}{8} = -0.1450 \\
x_3^{(2)} &= -\tfrac{1}{8}(0.1200) & &+ \tfrac{1}{8}(-0.1450) & & & &- \tfrac{3}{8} = -0.4081
\end{aligned}$$

and

$$\begin{aligned}
x_1^{(3)} &= & & - \tfrac{2}{10}(-0.1450) & &+ \tfrac{1}{10}(-0.4081) & &+ \tfrac{1}{10} = 0.1118 \\
x_2^{(3)} &= \tfrac{2}{8}(0.1118) & & & &+ \tfrac{1}{8}(-0.4081) & &- \tfrac{1}{8} = -0.1481 \\
x_3^{(3)} &= -\tfrac{1}{8}(0.1118) & &+ \tfrac{1}{8}(-0.1481) & & & &- \tfrac{3}{8} = -0.4075
\end{aligned}$$

Repeated application yields:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0.1000	−0.1000	−0.4000
2	0.1200	−0.1450	−0.4081
3	0.1118	−0.1481	−0.4075
4	0.1111	−0.1482	−0.4074
5	0.1111	−0.1482	−0.4074

So the method converges (to four significant digits) in four iterations.

5. The order in which equations are written can affect whether or not they can be solved, in their original form, by iterative methods. For example, consider

$$\begin{array}{ccccc} 4x_1 & + & x_2 & = & 1 \\ x_1 & + & 4x_2 & = & 1 \end{array} \quad \text{and} \quad \begin{array}{ccccc} x_1 & + & 4x_2 & = & 1 \\ 4x_1 & + & x_2 & = & 1 \end{array}$$

Show that the first of these is solvable by either the Jacobi or Gauss-Seidel method, but that not only is the second not solvable by either, but the Gauss-Seidel algorithm diverges faster than Jacobi for that system.

solution:

The Jacobi algorithm for the system in the original form is:

$$\begin{array}{ccccccc} x_1 & = & & - & \frac{1}{4}x_2 & + & \frac{1}{4} \\ x_2 & = & -\frac{1}{4}x_1 & & & + & \frac{1}{4} \end{array}$$

So the Jacobi algorithm for this problem becomes:

$$\begin{array}{ccccccc} x_1^{(k+1)} & = & & - & \frac{1}{4}x_2^{(k)} & + & \frac{1}{4} \\ x_2^{(k+1)} & = & -\frac{1}{4}x_1^{(k)} & & & + & \frac{1}{4} \end{array}$$

or, in matrix-vector form:

$$\mathbf{x}^{(k+1)} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix}}_{\mathbf{G}_{\text{Jacobi}}} \mathbf{x}^{(k)} + \underbrace{\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}}_{\mathbf{b}'_{\text{Jacobi}}}$$

Clearly, $\| \mathbf{G}_{\text{Jacobi}} \|_\infty = \dfrac{1}{4} < 1$ and therefore convergence is guaranteed. Similarly, the Gauss-Seidel form of the original problem is:

$$\begin{array}{ccccccc} x_1^{(k+1)} & & & = & - & \frac{1}{4}x_2^{(k)} & + & \frac{1}{4} \\ \frac{1}{4}x_1^{(k+1)} & + & x_2^{(k+1)} & = & & & + & \frac{1}{4} \end{array}$$

In matrix-vector form, this is:

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix} \mathbf{x}^{(k+1)} = \begin{bmatrix} 0 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

solution:

or equivalently:

$$\mathbf{x}^{(k+1)} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix}}_{\mathbf{G}_{\text{GS}}} \mathbf{x}^{(k)} \; + \; \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}}_{\mathbf{b}'_{\text{GS}}}$$

Direct computation will verify that:

$$\mathbf{G}_{\text{GS}} = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} \\ 0 & \frac{1}{16} \end{bmatrix}$$

and since clearly, $\| \mathbf{G}_{\text{GS}} \|_\infty = \dfrac{1}{4} < 1$, therefore convergence is again guaranteed.

For the rewritten system:

$$\begin{array}{rcrcl} x_1 & + & 4x_2 & = & 1 \\ 4x_1 & + & x_2 & = & 1 \end{array}$$

the equivalent Jacobi formulation is:

$$\begin{array}{rcrrrrl} x_1^{(k+1)} & = & & & - & 4x_2^{(k)} & + \; 1 \\ x_2^{(k+1)} & = & - & 4x_1^{(k)} & & & + \; 1 \end{array}$$

or, in matrix-vector form:

$$\mathbf{x}^{(k+1)} = \underbrace{\begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}}_{\hat{\mathbf{G}}_{\text{Jacobi}}} \mathbf{x}^{(k)} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\hat{\mathbf{b}}'_{\text{Jacobi}}}$$

Direct computation can show that the eigenvalues of $\hat{\mathbf{G}}_{\text{Jacobi}}$ are $\lambda = \pm 4$. Therefore, the associated spectral radius is:

$$\rho_{\text{Jacobi}} = \max_i |\lambda_i| = 4 > 1$$

Hence, the Jacobi method must diverge.

solution:

The Gauss-Seidel formulation of the rewritten system is:

$$\begin{array}{rcrcl} x_1^{(k+1)} & & & = & - \; 4x_2^{(k)} \; + \; 1 \\ 4x_1^{(k+1)} & + & x_2^{(k+1)} & = & \phantom{- \; 4x_2^{(k)}} \; + \; 1 \end{array}$$

or, in matrix-vector form:

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \mathbf{x}^{(k+1)} = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or, equivalently

$$\mathbf{x}^{(k+1)} = \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}}_{\hat{\mathbf{G}}_{\text{GS}}} \mathbf{x}^{(k)} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\hat{\mathbf{b}}'_{\text{GS}}}$$

Direct computation can show that

$$\hat{\mathbf{G}}_{\text{GS}} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 0 & 16 \end{bmatrix}$$

and that its eigenvalues are $\lambda = 0, 16$. Therefore, the associated spectral radius is:

$$\rho_{\text{GS}} = \max_i |\lambda_i| = 16 > 1$$

Hence, the Gauss-Seidel method must diverge. Moreover, since its spectral radius is larger than the spectral radius of the associated Jacobi method, it will diverge more rapidly.

6. As discussed, diagonal dominance is a sufficient, but not necessary condition for convergence of an iterative method. Show that both the Jacobi and Gauss-Seidel methods converge for the system:

$$\begin{array}{ccccc} x_1 & + & 2x_2 & = & 1 \\ x_1 & + & 10x_2 & = & -2 \end{array}$$

even though the system is clearly not diagonally dominant.

solution:

The Jacobi algorithm for the system in the original form is:

$$\begin{array}{cccccc} x_1 & = & & - & 2x_2 & + & 1 \\ x_2 & = & -\frac{1}{10}x_1 & & & - & \frac{2}{10} \end{array}$$

So the Jacobi algorithm for this problem becomes:

$$\begin{array}{cccccc} x_1^{(k+1)} & = & & - & 2x_2^{(k)} & + & 1 \\ x_2^{(k+1)} & = & -\frac{1}{10}x_1^{(k)} & & & - & \frac{2}{10} \end{array}$$

or, in matrix-vector form:

$$\mathbf{x}^{(k+1)} = \underbrace{\begin{bmatrix} 0 & -2 \\ -\frac{1}{10} & 0 \end{bmatrix}}_{\mathbf{G}_{\text{Jacobi}}} \mathbf{x}^{(k)} + \underbrace{\begin{bmatrix} 1 \\ -\frac{2}{10} \end{bmatrix}}_{\mathbf{b}'_{\text{Jacobi}}}$$

While $\| \mathbf{G}_{\text{Jacobi}} \|_\infty = 2 > 1$, unfortunately, the norm test is a only sufficient condition. Therefore, here we must check the eigenvalues of $\mathbf{G}_{\text{Jacobi}}$. These are easily shown to be $\lambda \doteq \pm 0.4472$. Therefore:

$$\rho_{\text{Jacobi}} = \max_i |\lambda_i| \doteq 0.4472 < 1$$

and so convergence is guaranteed, even though the matrix was not diagonally dominant. Similarly, the Gauss-Seidel form of the original problem is:

$$\begin{array}{cccccccc} x_1^{(k+1)} & & & = & - & 2x_2^{(k)} & + & 1 \\ \frac{1}{10}x_1^{(k+1)} & + & x_2^{(k+1)} & = & & & - & \frac{2}{10} \end{array}$$

solution:

In matrix-vector form, this is:

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{10} & 1 \end{bmatrix} \mathbf{x}^{(k+1)} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \begin{bmatrix} 1 \\ \frac{2}{10} \end{bmatrix}$$

or equivalently:

$$\mathbf{x}^{(k+1)} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{10} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}}_{\mathbf{G}_{\mathrm{GS}}} \mathbf{x}^{(k)} + \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{10} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2}{10} \end{bmatrix}}_{\mathbf{b}'_{\mathrm{GS}}}$$

Direct computation will verify that:

$$\mathbf{G}_{\mathrm{GS}} = \begin{bmatrix} 1 & 0 \\ \frac{1}{10} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & \frac{2}{10} \end{bmatrix}$$

Since clearly the eigenvalues of $\mathbf{G}_{\mathrm{GS}}$, are $\lambda = 0, 0.2$, therefore

$$\rho_{\mathrm{GS}} = \max_i |\lambda_i| = 0.2 < 1$$

and convergence is again guaranteed. Also, since

$$\rho_{\mathrm{GS}} < \rho_{\mathrm{Jacobi}}$$

the Gauss-Seidel method will converge more rapidly here.